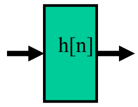


Discrete
Time System

Applications of Z-Transform in Signal Processing - I

Yogananda Isukapalli

Causal and Stable systems

Definition:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

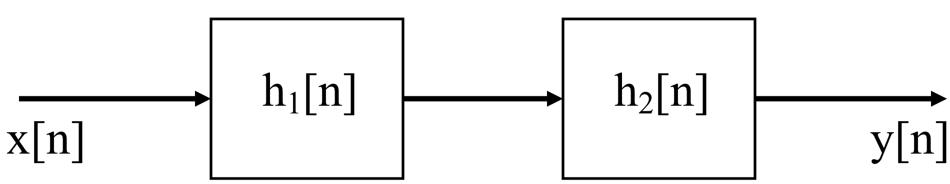
Causal : $\{h[n]\} = 0$ for $n < 0$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

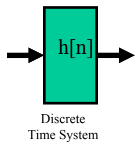
Example :

Consider two identical systems :

$\{h[n]\} = (1/2)^n$, $n \geq 0$



$h_1[n] = h_2[n] = (1/2)^n$, $n \geq 0$



Response :

$$y[n] = (x[n] * h_1[n]) * h_2[n]$$

Taking the z transform we have:

$$Y(z) = X(z) H_1(z) H_2(z)$$

Now:

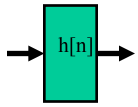
$$H_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$H_2(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$Y(z) = \frac{X(z)}{\left(1 - \frac{1}{2} z^{-1}\right)^2}$$

Suppose Now : $x[n] = 1$ for $n \geq 0$

$$X(z) = \frac{1}{1 - z^{-1}}$$



Discrete
Time System

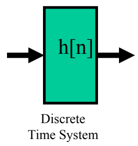
Therefore now the response of the system for the given Input would be :

$$Y(z) = \frac{1}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

$$i.e. \quad Y(z) = \frac{z^3}{(z-1)(z-\frac{1}{2})^2}$$

$$i.e. \; Y(z) = \frac{4z}{z-1} - \frac{2z}{(z-\frac{1}{2})} - \frac{z^2}{(z-\frac{1}{2})^2}$$

$$y[n] = 4 - 2(\frac{1}{2})^n - (n+1)(\frac{1}{2})^n .. n \geq 0$$



Stability Theorem :

The system is said to be stable if and only if :

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Recall :

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

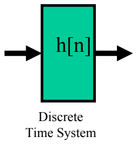
$$assume:$$

$$|x[.]| < l < \infty$$

$$then$$

$$|y[n]| \le \sum_{m=-\infty}^{\infty} |h[m]||x[n-m]|$$

$$|y[n]| \le l \sum_{m=-\infty}^{\infty} |h[m]|$$



Follows : |y[n]| is bounded if and only if :

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Stable Causal Systems :

Example :

$$\{h[n]\} \leftrightarrow \{H(z)\}$$

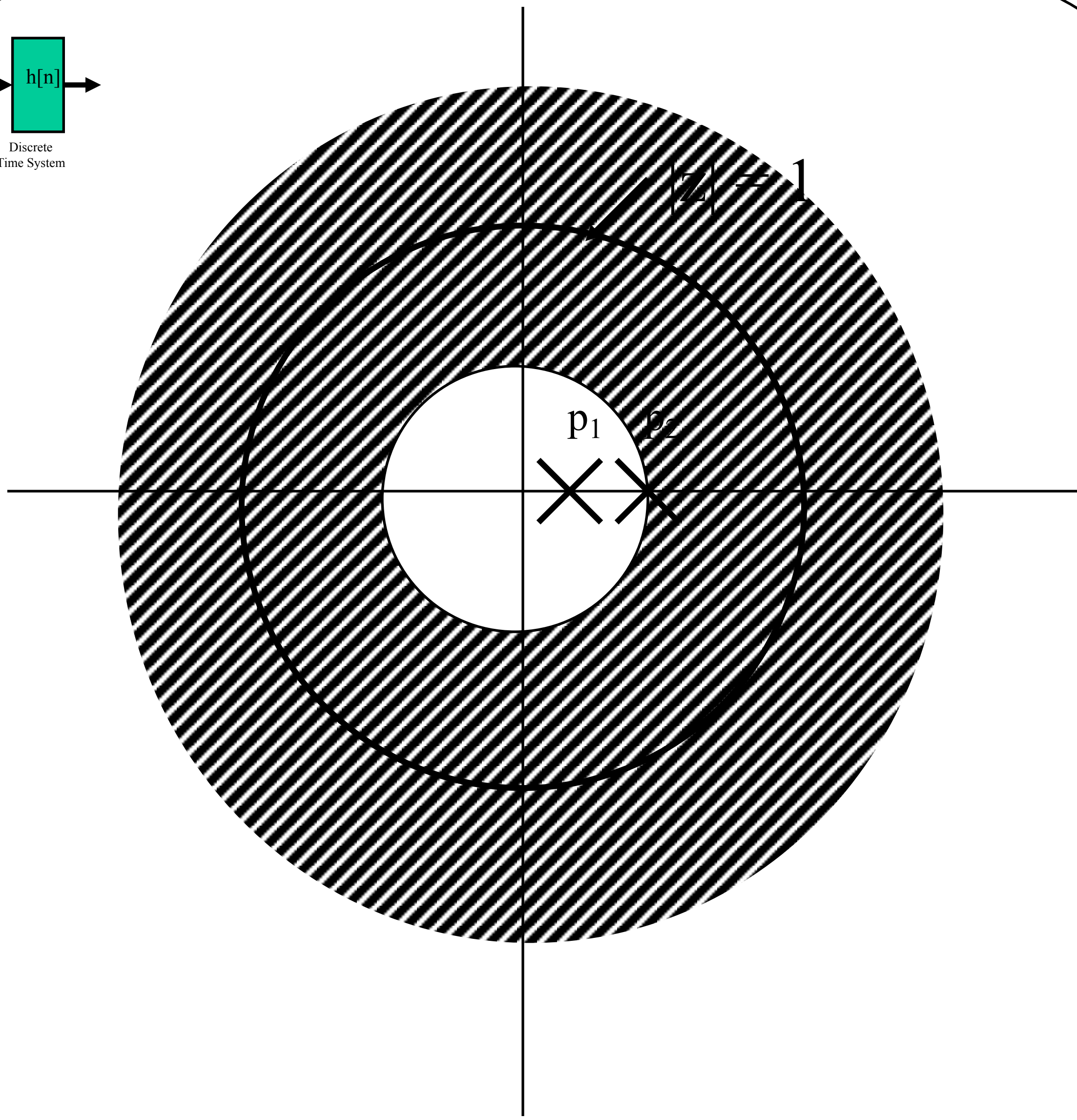
$$H(z) = C_0 + \frac{C_1}{z - p_1} + \frac{C_2}{z - p_2}$$

Poles : p_1, p_2

Follows : $h[n] = C_0\delta[n] + C_1 p_1^n + C_2 p_2^n$

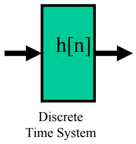
If $h[n] \to 0$, as $n \to \infty$ then : $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Implies $\to$ if $|p_i| < 1$ for all i
$h[n] \to 0$, as $n \to \infty$



ROC includes the unit circle $|z| = 1$ for a stable $\{|p_i| < 1\}$ causal systems.

Formal Definition: A causal system with a rational transfer function H(z) is stable if all the poles (p_i) of H(z) are located inside the unit circle.



Example :

$$H(z) = \frac{z}{(z-p)^2}$$

$$h[n] = np^{n-1}u[n]$$

$$h[n] \to 0 \quad \text{if } |p| < 1$$

as $n \to \infty$

Stable even for repeated poles if $|p_i| < 1$

<u>Marginally Stable</u>

$$H(z) = \frac{z}{(z-1)}$$

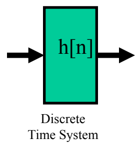
$$h[n] = 1^n$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

is not satisfied.

For finite N $\sum_{n=-\infty}^{\infty} |h[n]|$ is bounded

pole is on $|z| = 1$



Complex Conjugate poles on the unit circle :

$$H(z) = \frac{1}{(1 - z^{-1} e^{j\theta_0})} + \frac{1}{(1 - z^{-1} e^{-j\theta_0})}$$

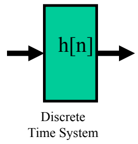
$$h[n] = e^{j\theta_0 n} + e^{-j\theta_0 n}$$

$$h[n] = 2\cos(\theta_0 n) u[n]$$

$$\sum | h[n] | < \infty \text{ is satisfied for finite N.}$$

Multiple poles repeated on the Unit circle will make the system to be unstable.

Note: **θ=*w*T (rad)**



Examples

1. The trapezoidal integration formula can be represented as an IIR digital filter represented by a difference equation given by

   $$y[n] = y[n-1] + (1/2)\{x[n] + x[n-1]\}$$

   with y[-1] = 0. Determine the transfer function of the above filter.

Soln) Given, the difference equation representing trapezoidal integration formula as:

$$\mathbf{y[n] = y[n-1] + (1/2)\{x[n] + x[n-1]\}}$$

Taking the z-transform of the above equation gives that:

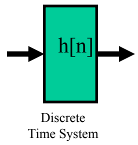
$$Y(z) = z^{-1}Y(z) + \frac{1}{2}\{\ X(z) + z^{-1}X(z)\ \}$$

$$i.e.\quad Y(z)[1-z^{-1}] = \frac{1}{2}\mathrm{X(z)}[1+z^{-1}]$$

Therefore, transfer function H(z) is :

$$\mathrm{H(z)} = \frac{\mathrm{Y(z)}}{\mathrm{X(z)}}$$

$$i.e.\quad H(z) = \frac{(1+z^{-1})}{2(1-z^{-1})}$$



2. Let $H(z)$ be the transfer function of a causal stable LTI discrete-time system. Let $G(z)$ be the transfer function obtained by replacing z^{-1} in $H(z)$ with $\alpha + z^{-1} / 1 + \alpha z^{-1}$. Show that $G(1)=H(1)$ and $G(-1)=H(-1)$.

Soln) Given that the transfer function G(z) is obtained from H(z) by replacing

$$z^{-1} \text{ by } \frac{\alpha + z^{-1}}{1+\alpha z^{-1}} \quad \text{i.e. } G(Z) = H(z)\Big|z^{-1} = \frac{\alpha + z^{-1}}{1+\alpha z^{-1}}$$

$$\text{i.e. replace z by } \frac{1+\alpha z^{-1}}{\alpha + z^{-1}}$$

or

$$\text{replace z by } \frac{\alpha + z}{1+\alpha z}$$

a) To show G(1) = H(1)

$$G(Z) = H(z)\Big|z = \frac{\alpha + z}{1+\alpha z}$$

When $z = 1$,

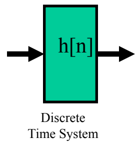
$$G(1) = H\left(\frac{\alpha + 1}{1+\alpha}\right) = H(1)$$

b) G(-1) = H(-1)

When $z = -1$,

$$G(-1) = H\left(\frac{\alpha - 1}{1-\alpha}\right)$$

$$= H(-1)$$



3. Determine the transfer function of a causal stable LTI discrete-time system described by the following difference equation:

y[n] = 5x[n] + 5x[n-1] + 0.4x[n-2] + 0.32x[n-3]
- 0.5y[n-1] + 0.34y[n-2] + 0.08y[n-3]

Express the transfer function in a factored form and sketch its pole-zero plot. Is the system BIBO stable?

Soln) The difference equation of a casual LTI discrete system is:

y[n] = 5x[n] + 5x[n-1] + 0.4x[n-2] + 0.32x[n-3]
- 0.5y[n-1] + 0.34y[n-2] + 0.08y[n-3]

Taking z-transform of the above equation:

$$Y(z) = 5X(z) + 5z^{-1}X(z) + 0.4z^{-2}X(z) + 0.32z^{-3}X(z)$$
$$-0.5z^{-1}Y(z) + 0.34z^{-2}Y(z) + 0.08z^{-3}Y(z)$$

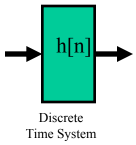
i.e.

$$\mathrm{Y(z)}[1+0.5z^{-1} - 0.34z^{-2} - 0.08z^{-3}] = X(z)[5+5z^{-1}+0.4z^{-2}+0.32z^{-3}]$$

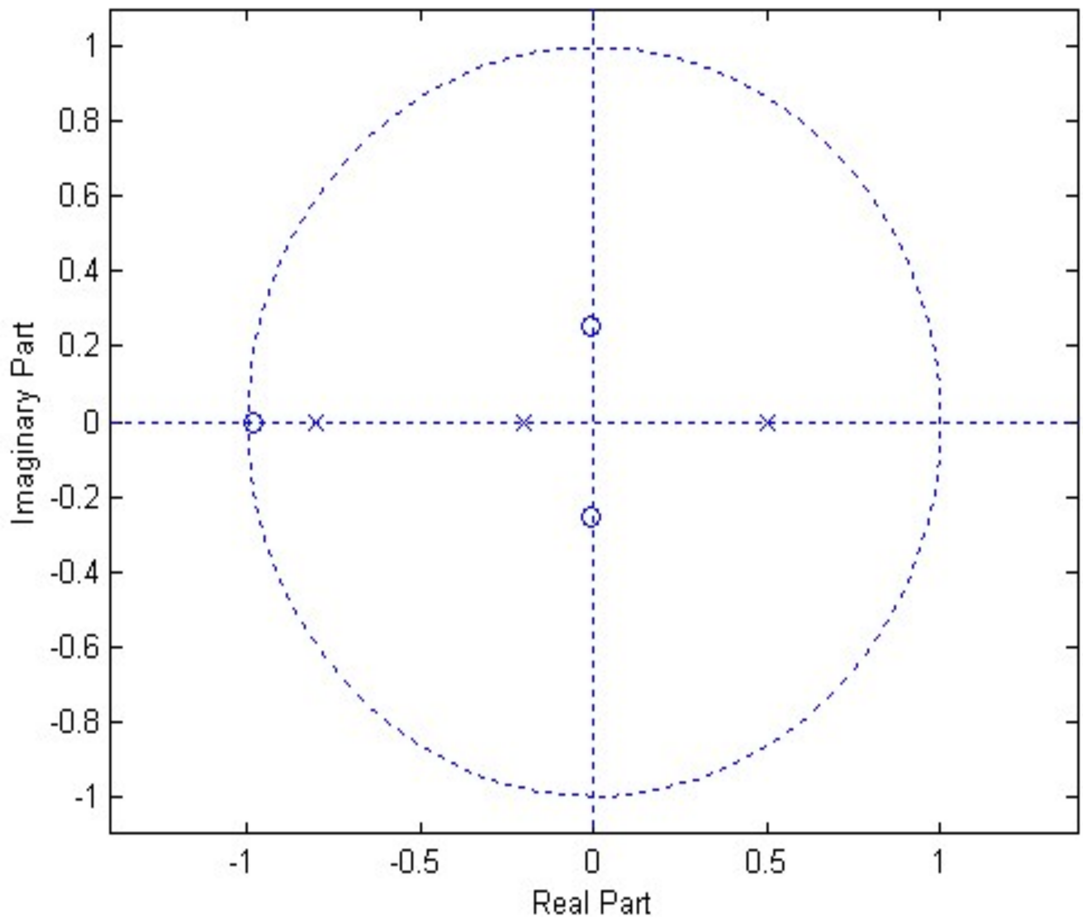
The transfer function H(z) is :

$$H(z) = \frac{\mathrm{Y(z)}}{\mathrm{X(z)}} = \frac{5+5z^{-1}+0.4z^{-2}+0.32z^{-3}}{1+0.5z^{-1}-0.34z^{-2}-0.08z^{-3}}$$

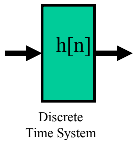
$$i.e. H(z) = \frac{-1.0256}{(1-0.8z^{-1})} + \frac{5.6237}{(1-0.5z^{-1})} + \frac{4.7619}{(1+0.2^{-1})} - 4$$



The pole zero plot of the transfer function using Matlab:



The system is stable since all the poles lie <u>inside</u> the unit circle



4. Using z-transform methods, determine the explicit expression For the impulse response h(n) of a causal LTI discrete-time system which develops an output $y[n]=4(0.75)^n \mu[n]$ for an input $x[n]=3(0.25)^n \mu[n]$.

Soln)

Given that the system response $\mathrm{y[n]} = 4(0.75)^{\mathrm{n}} \mu[\mathrm{n}]$

for an input $\mathrm{x[n]} = 3(0.25)^{\mathrm{n}} \mu[\mathrm{n}]$

Taking z transform of each of the equation, we get

$$y[\mathrm{z}] = 4\frac{1}{(1-0.75z^{-1})} \quad \text{and} \quad \mathrm{X}(z) = 3\frac{1}{(1-0.25z^{-1})}$$

Therefore, the transfer function H(z) is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4}{3}.\frac{(1-0.25z^{-1})}{(1-0.75z^{-1})}$$

$$= \frac{4}{3}\left[\frac{1}{(1-0.75z^{-1})} - \frac{0.25z^{-1}}{(1-0.75z^{-1})}\right]$$

$$\mathrm{h[n]} = \frac{4}{3}\left[\ (0.75)^{\mathrm{n}} \mu[n] - 0.25(0.75)^{\mathrm{n-1}} \mu[n-1]\ \right]$$